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Linear Algebra and its Applications 388 (2004) 279–288

LINEAR ALGEBRA AND ITS APPLICATIONS

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The difference and sum of projectors

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Received 18 February 2002; accepted 5 March 2004

Submitted by G.P.H. Styan

Abstract

The aim of this paper is to present new results on the invertibility of the sum of projectors, and new relations between the nonsingularity of the difference and the sum of projectors. We also give simple proofs, without referring to rank theory, of some results on the nonsingularity of the difference of projectors obtained by Groß and Trenkler [SIAM J. Matrix Anal. Appl. 21 (1999) 390].

AMS classification: 15A09; 15A24; 15A57

Keywords: Projector; Direct sum

1. The nonsingularity of the difference of projectors

In [3], Groß and Trenkler consider the nonsingularity of $P - Q$ for general matrix projectors. Their methods employ relations for the ranks of matrices developed by Marsaglia and Styan [7], and the authors obtain some informative rank identities. In the present paper we provide new simple proofs, without reference to rank theory,

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1 Partially suported by the University of Melbourne Visiting Scholar Grant 2001.

2 Partially supported by the Grant Agency of Czech Republic, Grant 201/02/0854.

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doi:10.1016/j.laa.2004.03.008

of some of their results relating to the invertibility of $P - Q$. We also obtain a new characterization of the nonsingularity of $P - Q$ in terms of the nonsingularity of $P + Q$. We give necessary and sufficient conditions for the nonsingularity of $P + Q$ and explicit equations for the inverse of $P + Q$ separately in the case when $P - Q$ is nonsingular, and when $P - Q$ is singular.

In a recent paper Tian and Styan [10] obtained many interesting equalities for the ranks of projector matrices and applied them to the invertibility of $P - Q$ and $P + Q$. Our approach is complementary to that of Groß and Trenkler [3] and Tian and Styan [10] as we primarily consider the kernel of a matrix to establish its nonsingularity.

We write $\mathscr{R}(A)$ and $\mathscr{N}(A)$ for the image and kernel of a matrix $A \in \mathbb{C}^{d\times d}$ and A^* for the conjugate transpose of A.

Lemma 1.1. *Let $A, B \in \mathbb{C}^{d\times d}$. Then*

$$\mathscr{R}(A^*) + \mathscr{R}(B^*) = \mathbb{C}^{d\times 1} \iff \mathscr{N}(A) \cap \mathscr{N}(B) = \{0\}, \tag{1.1}$$

$$\mathscr{R}(A^*) \cap \mathscr{R}(B^*) = \{0\} \iff \mathscr{N}(A) + \mathscr{N}(B) = \mathbb{C}^{d\times 1}. \tag{1.2}$$

Proof. Let $M^\perp$ denote the orthogonal complement of a subspace M of $\mathbb{C}^{d\times 1}$. From the identities

$$(M \cap N)^\perp = M^\perp + N^\perp, \qquad (M + N)^\perp = M^\perp \cap M^\perp$$

valid for any subspaces M, N of $\mathbb{C}^{d\times 1}$ and from the relation $\mathscr{R}(A^*)^\perp = \mathscr{N}(A)$ we obtain the result. $\square$

The equivalence of (i), (iii), (iv) and (v) of the next theorem was proved by Groß and Trenkler in [3, Corollaries 1 and 5] as a consequence of equalities for the matrix rank developed in [7] and refined in [3, Theorems 1–3]. We present a simple proof of the invertibility of $A \in \mathbb{C}^{d\times d}$ by showing that $\mathscr{N}(A) = \{0\}$, for the price of losing some information on rank obtained in [3].

A matrix $P \in \mathbb{C}^{d\times d}$ is a *projector* if $P^2 = P$; it is an *orthogonal projector* if, in addition, $P^* = P$.

Theorem 1.2. *Let $P, Q \in \mathbb{C}^{d\times d}$ be projectors. The following conditions are equivalent:*

(i) $\mathbb{C}^{d\times 1} = \mathscr{R}(P) \oplus \mathscr{R}(Q)$ *and* $\mathbb{C}^{d\times 1} = \mathscr{R}(P^*) \oplus \mathscr{R}(Q^*)$.
(ii) $\mathbb{C}^{d\times 1} = \mathscr{R}(P) \oplus \mathscr{R}(Q)$ *and* $\mathbb{C}^{d\times 1} = \mathscr{N}(P) \oplus \mathscr{N}(Q)$.
(iii) $\mathscr{R}(P) \cap \mathscr{R}(Q) = \{0\}$ *and* $\mathscr{N}(P) \cap \mathscr{N}(Q) = \{0\}$.
(iv) $P - Q$ *is nonsingular*.
(v) $I - PQ$ *and* $P + Q - PQ$ *are nonsingular*.

Proof. (i) $\Longleftrightarrow$ (ii). This follows from Lemma 1.1.

(ii) $\Longrightarrow$ (iii). This follows from the definition of the direct sum.

(iii) $\Longrightarrow$ (iv). Let $(P-Q)x=0$. Then $Px=Qx\in\mathscr{R}(P)\cap\mathscr{R}(Q)=\{0\}$, and $x\in\mathscr{N}(P)\cap\mathscr{N}(Q)=\{0\}$. Thus $\mathscr{N}(P-Q)=\{0\}$ and $P-Q$ is nonsingular.

(iv) $\Longrightarrow$ (v). We show that $\mathscr{N}(I-PQ)=\{0\}$. Let $(I-PQ)x=0$. Then $x=PQx=Px$, and $(P-Q)^2x=(I-PQ)x=0$. Since $(P-Q)^2$ is nonsingular, $x=0$. So $I-PQ$ is nonsingular. Similarly, from $(I-P)-(I-Q)=Q-P$ we deduce that $I-(I-P)(I-Q)=P+Q-PQ$ is nonsingular.

(v) $\Longrightarrow$ (ii). Let W be the inverse of $P+Q-PQ$. From $W(P+Q-PQ)=I$ we get $I=WP+W(I-P)Q$, that is, $\mathscr{N}(P)\cap\mathscr{N}(Q)=\{0\}$. The equation $(P+Q-PQ)W=I$ implies $I=P(I-Q)W+QW$, and $\mathbb{C}^{d\times 1}=\mathscr{R}(P)+\mathscr{R}(Q)$. Similarly, the nonsingularity of $I-PQ=(I-P)+(I-Q)-(I-P)(I-Q)$ yields $\mathscr{N}(I-P)\cap\mathscr{N}(I-Q)=\mathscr{R}(P)\cap\mathscr{R}(Q)=\{0\}$ and $\mathbb{C}^{d\times 1}=\mathscr{R}(I-P)+\mathscr{R}(I-Q)=\mathscr{N}(P)+\mathscr{N}(Q)$. Hence $\mathbb{C}^{d\times 1}=\mathscr{R}(P)\oplus\mathscr{R}(Q)=\mathscr{N}(P)\oplus\mathscr{N}(Q)$. □

Corollary 1.3. *Let $P, Q\in\mathbb{C}^{d\times d}$ be projectors. The following conditions are equivalent*:

(i) $\mathscr{R}(P)\oplus\mathscr{R}(Q)=\mathscr{N}(P)\oplus\mathscr{N}(Q)=\mathscr{N}(P)\oplus\mathscr{R}(Q)=\mathscr{R}(P)\oplus\mathscr{N}(Q)=\mathbb{C}^{d\times 1}$.

(ii) *$P-Q$ and $I-P-Q$ are nonsingular.*

(iii) *$PQ-QP$ is nonsingular.*

Proof. The equivalence of (i) and (ii) follows from Theorem 1.2 when we apply it first to P and Q, and then to $I-P$ and Q. For the equivalence of (ii) and (iii) we observe that

$$PQ-QP=(I-P-Q)(P-Q). \qquad \square$$

Tian and Styan in [10, Corollary 2.10] proved the equivalence of (ii) and (iii) of Corollary 1.3 as a consequence of rank equalitites. In place of our condition (i) they have

$$\mathbb{C}^{d\times 1}=\mathscr{R}(P)\oplus\mathscr{R}(Q)=\mathscr{R}(P^*)\oplus\mathscr{R}(Q^*) \text{ and}$$
$$\operatorname{rank}(PQ)=\operatorname{rank}(QP)=\operatorname{rank}(P)=\operatorname{rank}(Q).$$

The conditions for the nonsingularity of $P-Q$, where P, Q are orthogonal projectors, are far better known than for the case of general projectors. Conditions equivalent to $H=\mathscr{R}(P)\oplus\mathscr{R}(Q)$, where P, Q are orthogonal projectors on a Hilbert space H, were investigated in [6,8,11], and in a number of recent papers [1,2,4,5,9,12,13]. For orthogonal projectors conditions (i) and (ii) of the preceding theorem reduce to $\mathbb{C}^{d\times 1}=\mathscr{R}(P)\oplus\mathscr{R}(Q)$ and $\mathbb{C}^{d\times 1}=\mathscr{N}(P)\oplus\mathscr{N}(Q)$, respectively.

2. Formulae for $(P - Q)^{-1}$ and $(P + Q)^{-1}$ when $P - Q$ is nonsingular

First we obtain a new characterization of the nonsingularity of $P - Q$ in terms of the nonsingularity of $P + Q$ and $I - PQ$.

Theorem 2.1. *Let $P, Q \in \mathbb{C}^{d\times d}$ be projectors. The following conditions are equivalent:*

(i) *$P - Q$ is nonsingular.*
(ii) *$P + Q$ and $I - PQ$ are nonsingular.*

Proof. (i) $\Longrightarrow$ (ii). If $(P + Q)x = 0$, then $Px = -Qx \in \mathscr{R}(P) \cap \mathscr{R}(Q) = \{0\}$ by Theorem 1.2 (iii), and $x \in \mathscr{N}(P) \cap \mathscr{N}(Q) = \{0\}$ again by Theorem 1.2 (iii). Hence $\mathscr{N}(P + Q) = \{0\}$ and $P + Q$ is nonsingular. The nonsingularity of $I - PQ$ follows from Theorem 1.2 (v).

(ii) $\Longrightarrow$ (i). Let $(P - Q)x = 0$, so that $Px = Qx = QPx = PQPx$. Then

$$
\begin{aligned}
Px &= (I - PQ)^{-1}(I - PQ)Px = (I - PQ)^{-1}(Px - PQPx) = 0,\\
(I - P)x &= (P + Q)^{-1}(P + Q)(I - P)x = (P + Q)^{-1}(Qx - QPx) = 0,
\end{aligned}
$$

and $x = Px + (I - P)x = 0$. Hence $\mathscr{N}(P - Q) = \{0\}$. $\square$

When P, Q are projectors such that $P - Q$ is nonsingular, we can use the projectors associated with the space decompositions

$$\mathbb{C}^{d\times 1} = \mathscr{R}(P) \oplus \mathscr{R}(Q) = \mathscr{N}(P) \oplus \mathscr{N}(Q)$$

to give explicit formulae for $(P - Q)^{-1}$ and $(P + Q)^{-1}$. To this end we define

$$F = P(P - Q)^{-1} = (P - Q)^{-1}(I - Q), \tag{2.1}$$
$$G = (P - Q)^{-1}P = (I - Q)(P - Q)^{-1} \tag{2.2}$$

(the alternative formulations are easy to establish). Both F and G are projectors. As a sample we prove $F^2 = F$:

$$
\begin{aligned}
F^2 &= (P - Q)^{-1}(I - Q)P(P - Q)^{-1}\\
&= (P - Q)^{-1}(I - Q)(P - Q)(P - Q)^{-1}\\
&= (P - Q)^{-1}(I - Q) = F.
\end{aligned}
$$

From the definitions we observe that

$$
\begin{aligned}
&\mathscr{R}(F) = \mathscr{R}(P), \qquad \mathscr{N}(F) = \mathscr{N}(I - Q) = \mathscr{R}(Q),\\
&\mathscr{R}(G) = \mathscr{R}(I - Q) = \mathscr{N}(Q), \qquad \mathscr{N}(G) = \mathscr{N}(P).
\end{aligned}
$$

It is convenient to write $P_{M,N}$ for the projector with the image M and kernel N.

We can now give explicit formulae for the inverses of the difference and sum of two projectors.

Theorem 2.2. *Let $P, Q \in \mathbb{C}^{d\times d}$ be projectors such that $P - Q$ is nonsingular, and let F, G be the projectors $F = P_{\mathscr{R}(P),\mathscr{R}(Q)}$, $G = P_{\mathscr{N}(Q),\mathscr{N}(P)}$, whose existence is guaranteed by Theorem* 1.2. *Then*

$$(P - Q)^{-1} = F - (I - G), \tag{2.3}$$
$$(P + Q)^{-1} = I - (I - G)F - G(I - F), \tag{2.4}$$
$$(P - Q)^{-1} = (P + Q)^{-1}(P - Q)(P + Q)^{-1}, \tag{2.5}$$
$$(P + Q)^{-1} = (P - Q)^{-1}(P + Q)(P - Q)^{-1}. \tag{2.6}$$

Proof. Projectors F and G are given by (2.1) and (2.2), respectively. From these equations we deduce that

$$FP = P, \quad PF = F, \quad FQ = 0, \quad QF = Q + F - I,$$
$$GP = G, \quad PG = P, \quad GQ = Q + G - I, \quad QG = 0.$$

After simple calculations we obtain

$$(P - Q)(F + G - I) = I,$$
$$(P + Q)(I - F - G + 2GF) = I,$$

from which (2.3) and (2.4) follow. Eqs. (2.5) and (2.6) are obtained from

$$(P + Q)(F + G - I)(P + Q) = P - Q,$$
$$(P - Q)(I - F - G + 2GF)(P - Q) = P + Q. \qquad \square \tag{2.7}$$

Formulae (2.3) and (2.4) can be rewritten in the following more transparent form:

$$(P - Q)^{-1} = P_{\mathscr{R}(P),\mathscr{R}(Q)} - P_{\mathscr{N}(P),\mathscr{N}(Q)}, \tag{2.8}$$
$$(P + Q)^{-1} = I - P_{\mathscr{N}(P),\mathscr{N}(Q)}P_{\mathscr{R}(P),\mathscr{R}(Q)} - P_{\mathscr{N}(Q),\mathscr{N}(P)}P_{\mathscr{R}(Q),\mathscr{R}(P)}. \tag{2.9}$$

3. Characterizing the nonsingularity of $P + Q$ when $P - Q$ is singular

Let P, Q be again projectors in $\mathbb{C}^{d\times d}$. The invertibility of $P - Q$ is sufficient, but not necessary, for the invertibility of $P + Q$ as demonstrated in the following example.

Example 3.1. Define the projectors

$$P=\begin{bmatrix}0&1&0\\0&1&0\\0&0&1\end{bmatrix},\qquad Q=\begin{bmatrix}0&0&0\\1&1&0\\0&0&1\end{bmatrix}.$$

Then

$$P+Q=\begin{bmatrix}0&1&0\\1&2&0\\0&0&2\end{bmatrix},\qquad P-Q=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix},$$

where $P+Q$ is nonsingular and $P-Q$ singular.

We observe that the inverse of $P+Q$ cannot be calculated from Eq. (2.4) (and, consequently (2.9)) if $P-Q$ is singular. Indeed, Eq. (2.9) implies the existence of direct sums $\mathbb{C}^{d\times 1}=\mathscr{R}(P)\oplus\mathscr{R}(Q)=\mathscr{N}(P)\oplus\mathscr{N}(Q)$, which in turn implies the invertibility of $P-Q$ by Theorem 1.2.

We give a new characterization of the invertibility of $P+Q$ in the following theorem.

Theorem 3.2. *Let $P, Q\in\mathbb{C}^{d\times d}$ be projectors. The following conditions are equivalent*:

(i) *$P+Q$ is nonsingular.*
(ii) *$\mathscr{R}(Q(I-P))\cap\mathscr{R}(P)=\{0\}$ and $\mathscr{N}((I-P)Q)\cap\mathscr{N}(P)=\{0\}$.*
(iii) *$\mathscr{R}(Q(I-P))+\mathscr{R}(P)=\mathbb{C}^{d\times 1}$ and $\mathscr{N}((I-P)Q)+\mathscr{N}(P)=\mathbb{C}^{d\times 1}$.*

Proof. (i) $\Longrightarrow$ (ii). Let $x\in\mathscr{R}(Q(I-P))\cap\mathscr{R}(P)$. Then $x=Px=Qx=Q(I-P)u$ for some $u\in\mathbb{C}^{d\times 1}$ and

$$(P+Q)x=2x=2Q(I-P)u=(P+Q)(I-P)(2u).$$

From the nonsingularity of $P+Q$ it follows that $x=(I-P)(2u)$, and $x=Px=0$. This implies $\mathscr{R}(Q(I-P))\cap\mathscr{R}(P)=\{0\}$. Further, let $(I-P)Qx=0$ and $Px=0$. Define $y=(P+Q)x$. Then $Qy=y$, $Py=y$, and

$$(P+Q)y=2y=2(P+Q)x;$$

since $P+Q$ is nonsingular, $x=\frac{1}{2}y$. Hence $Qx=\frac{1}{2}Qy=\frac{1}{2}y=x$,

$$x=\frac{1}{2}y=\frac{1}{2}(P+Q)x=\frac{1}{2}x,$$

and $x=0$. This implies $\mathscr{N}((I-P)Q)\cap\mathscr{N}(P)=\{0\}$, and (ii) holds.

(ii) $\Longrightarrow$ (i). Let $(P+Q)x=0$. Then $Px=-Qx=QPx$, and

$$Px=-QPx-Q(I-P)x=-Px-Q(I-P)x,$$

which gives $Q(I-P)(-x)=2Px\in\mathscr{R}(Q(I-P))\cap\mathscr{R}(P)=\{0\}$. Hence $Px=0=Qx$, and $x\in\mathscr{N}((I-P)Q)\cap\mathscr{N}(P)$, which gives $x=0$. This proves that $\mathscr{N}(P+Q)=\{0\}$, and the result follows.

To prove the equivalence of (i) and (iii), we note that $P+Q$ is nonsingular if and only if P^*+Q^* is, then use the equivalence of (i) and (ii) with P^*, Q^* in place of P, Q, respectively, and apply Lemma 1.1. $\square$

Groß and Trenkler [3, Corollary 2] proved that $P+Q$ is nonsingular if and only if $\mathscr{R}(Q(I-P))\cap\mathscr{R}(P)=\{0\}$ and $\mathscr{N}(Q)\cap\mathscr{N}(P)=\{0\}$. This result is recovered from the equivalence of (i) and (ii) in the preceding theorem by referring to the facts that $\mathscr{N}(Q)\cap\mathscr{N}(P)\subset\mathscr{N}((I-P)Q)\cap\mathscr{N}(P)$ and that the converse inclusion holds in every case where $\mathscr{R}(Q(I-P))\cap\mathscr{R}(P)=\{0\}$.

From Theorem 3.2 we obtain immediately the following result.

Theorem 3.3. *Let $P, Q\in\mathbb{C}^{d\times d}$ be projectors. Then $P+Q$ is nonsingular if and only if*

$$\mathscr{N}((I-P)Q)\oplus\mathscr{N}(P)=\mathbb{C}^{d\times 1}\quad\text{and}\quad\mathscr{R}(Q(I-P))\oplus\mathscr{R}(P)=\mathbb{C}^{d\times 1}. \tag{3.1}$$

Remark 3.4. Since the condition '$P+Q$ is nonsingular' is symmetrical in P and Q, we can obtain counterparts of the necessary and sufficient conditions in Theorems 3.2 and 3.3 by interchanging P and Q.

In [10, Corollary 2.5] Tian and Styan obtained interesting conditions for the invertibility of $P+Q$ involving rank arguments and relations between images of block matrices. The equivalence of (ii) and (iii) in the following corollary coincides with their [10, Corollary 2.13].

Corollary 3.5. *Let P, Q be projectors. The following conditions are equivalent:*

(i) $\mathbb{C}^{d\times 1}=\mathscr{N}(P)\oplus\mathscr{R}(Q)=\mathscr{R}(P)\oplus\mathscr{N}(Q)$ *and* $\mathscr{N}((I-P)Q)\cap\mathscr{N}(P)=\mathscr{R}(Q(I-P))\cap\mathscr{R}(P)=\{0\}$.
(ii) *$P+Q$ and $I-P-Q$ are nonsingular.*
(iii) *$PQ+QP$ is nonsingular.*

Proof. From Theorem 1.2 applied to $I-P$ and Q and from Theorem 3.2 we obtain the equivalence of (i) and (ii). The equivalence of (ii) and (iii) follows from $(I-P-Q)(P+Q)=-(PQ+QP)$. $\square$

4. Formulae for $(P+Q)^{-1}$ when $P-Q$ is singular

In this section we again assume that P, Q are projectors in $\mathbb{C}^{d\times d}$ such that $P+Q$ is nonsingular, but $P-Q$ singular. Eq. (2.9) cannot be used to describe $(P+Q)^{-1}$ since the projectors involved in that formula need not exist. We base our construction on yet another decomposition of $\mathbb{C}^{d\times 1}$ valid when $P+Q$ is nonsingular, and on our results of Section 2 and 3.

Theorem 4.1. *Let $P, Q \in \mathbb{C}^{d\times d}$ be projectors such that $P+Q$ is nonsingular. Then*

$$\mathbb{C}^{d\times 1} = \mathscr{R}(Q(I-P)) \oplus \mathscr{N}((I-P)Q), \tag{4.1}$$

and the inverse of $P+Q$ is given by the equation

$$(P+Q)^{-1} = \left(I - \frac{1}{2}P_{U,V}\right)\left(I - P_{\mathscr{N}(P),N}P_{\mathscr{R}(P),M} - P_{N,\mathscr{N}(P)}P_{M,\mathscr{R}(P)}\right), \tag{4.2}$$

where

$$M = \mathscr{R}(Q(I-P)), \qquad N = \mathscr{N}((I-P)Q), \tag{4.3}$$

$$U = \mathscr{R}(P) \cap \mathscr{R}(Q), \qquad V = \mathscr{R}(Q(I-P)) \oplus \mathscr{N}(Q). \tag{4.4}$$

Proof. Suppose that $x \in \mathscr{R}(Q(I-P)) \cap \mathscr{N}((I-P)Q)$. Then $x = Qx$ and $x = Px + (I-P)Qx = Px$. So $x \in \mathscr{R}(Q(I-P)) \cap \mathscr{R}(P) = \{0\}$ by (3.1). This proves that

$$\mathscr{R}(Q(I-P)) \cap \mathscr{N}((I-P)Q) = \{0\}. \tag{4.5}$$

Eq. (4.5) remains valid when we replace P, Q by P^*, Q^*, respectively. Applying Lemma 1.1 to this, we obtain

$$\mathscr{R}(Q(I-P)) + \mathscr{N}((I-P)Q) = \mathbb{C}^{d\times 1}. \tag{4.6}$$

Combination of (4.5) and (4.6) yields (4.1).

Let $S = P_{M,N}$. Since $P+Q$ is nonsingular, Theorem 3.2 implies

$$\mathscr{N}(S) \cap \mathscr{N}(P) = \{0\} \text{ and } \mathscr{R}(S) \cap \mathscr{R}(P) = \{0\},$$

that is,

$$\mathscr{N}((I-P)Q) \cap \mathscr{N}(P) = \{0\} \text{ and } \mathscr{R}(Q(I-P)) \cap \mathscr{R}(P) = \{0\}.$$

According to part (iii) $\Longleftrightarrow$ (iv) of Theorem 1.2, $P-S$ is nonsingular. Then so is $P+S$ by Theorem 2.1.

We can verify that

$$N = \mathscr{N}((I-P)Q) = \mathscr{N}(Q) \oplus (\mathscr{R}(P) \cap \mathscr{R}(Q)) = \mathscr{N}(Q) \oplus U.$$

From this decomposition and from (4.1) we obtain $\mathbb{C}^{d\times 1} = U \oplus V$. We define $T = P_{U,V}$, and verify that $T+S=Q$. First we note that $ST = P_{M,N}P_{U,V} = 0$ as $U \subset N$ and $TS = P_{U,V}P_{M,N} = 0$ as $M \subset V$. Then $(T+S)^2 = T+S$ is a projector with $\mathscr{N}(T+S) = \mathscr{N}(T) \cap \mathscr{N}(S)$ and $\mathscr{R}(T+S) = \mathscr{R}(T) \oplus \mathscr{R}(S)$. We have $\mathscr{N}(T+S) = V \cap N = \mathscr{N}(Q)$, $\mathscr{R}(T+S) = U \oplus M$, and

$$\mathbb{C}^{d\times 1} = \mathscr{N}(T+S) \oplus \mathscr{R}(T+S) = \mathscr{N}(Q) \oplus (U \oplus M).$$

Since $U \oplus M \subset \mathscr{R}(Q)$, we conclude that $\mathscr{R}(T+S) = \mathscr{R}(Q)$, that is, $T+S=Q$. We can then write

$$P+Q = P+S+T = (P+S)(I+(P+S)^{-1}T).$$

Thus $I+(P+S)^{-1}T$ is nonsingular, and

$$(P+Q)^{-1} = (I+(P+S)^{-1}T)^{-1}(P+S)^{-1}.$$

Since $P-S$ is nonsingular, the results of Section 2 supply the formula for the inverse of $P+S$. By Eq. (2.9),

$$(P+S)^{-1} = I - P_{\mathscr{N}(P),\mathscr{N}(S)}P_{\mathscr{R}(P),\mathscr{R}(S)} - P_{\mathscr{N}(S),\mathscr{N}(P)}P_{\mathscr{R}(S)\mathscr{R}(P)}. \tag{4.7}$$

To calculate $(I+(P+S)^{-1}T)^{-1}$ we note that $(P+S)T = PT+ST = PT = T$, the last equation valid in view of $\mathscr{R}(T) \subset \mathscr{R}(P)$. Hence $(P+S)^{-1}T = T$,

$$(I+(P+S)^{-1}T)^{-1} = (I+T)^{-1} = I - \frac{1}{2}T,$$

and

$$(P+Q)^{-1} = \left(I - \frac{1}{2}T\right)(P+S)^{-1}.$$

Equality (4.2) then follows from this expression and from (4.7). □

Example 4.2. Let P and Q be the projectors from Example 3.1. We have

$$Q(I-P) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (I-P)Q = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with $\mathscr{R}(Q(I-P)) = \operatorname{span}(e_2)$ and $\mathscr{N}((I-P)Q) = \operatorname{span}(e_1-e_2, e_3)$, where (e_1, e_2, e_3) is the standard basis of $\mathbb{C}^{3\times 1}$. Further, $\mathscr{N}(P) = \operatorname{span}(e_1)$, and $\mathscr{R}(P) \cap \mathscr{R}(Q) = \operatorname{span}(e_3)$. We calculate the projectors involved in Eq. (4.2):

$$P_{U,V} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{\mathscr{N}(P),N} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$P_{\mathscr{R}(P),M} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $P_{N,\mathscr{N}(P)} = I - P_{\mathscr{N}(P),N}$, $P_{M,\mathscr{R}(P)} = I - P_{\mathscr{R}(P),M}$. Then

$$\left(I - \frac{1}{2}P_{U,V}\right)\left(I - P_{\mathscr{N}(P),N}P_{\mathscr{R}(P),M} - P_{N,\mathscr{N}(P)}P_{M,\mathscr{R}(P)}\right) = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix},$$

which agrees with $(P + Q)^{-1}$.

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